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1993 J. Phys. A: Math. Gen. 26 L299

(http://iopscience.iop.org/0305-4470/26/6/002)

J. Phys. A: Math. Gen. 26 (1993) L299–L306. Printed in the UK

LETTER TO THE EDITOR

Sets of covariant and contravariant spinors for $SU_q(2)$ and alternative quantizations

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Received 27 May 1992

Abstract. m sets of covariant and contravariant q-bosonic spinors acting in the tensor product of m Fock spaces are constructed and their q-commutation relations determined. Both their transformation properties under the q-algebra $su_q(2)$ and its dual matrix pseudogroup $SU_q(2)$ are considered. It is proved that for $m > 1$ the contravariant spinors are not related to the covariant ones through Hermitian conjugation. Similar results are also obtained for q-fermionic spinors.

In recent years, it has been observed that essentially two types of q-deformed Heisenberg–Weyl and Clifford algebras are relevant to the theory of quantum groups and supergroups.

On one hand, Biedenharn (1989) and Macfarlane (1989) independently introduced a q-analogue of the harmonic oscillator and its associated q-boson operators. By considering two independent pairs of such operators $a_i^\dagger$, a_i, $i = 1, 2$, they extended the Schwinger realization of $su(2)$ to the corresponding q-algebra $su_q(2)$. The same type of construction was then carried through for most q-algebras (Sun and Fu 1989, Hayashi 1990, Kulish and Damaskinsky 1990) and, by introducing a q-deformed fermionic oscillator, for most q-superalgebras (Chaichian and Kulish 1990, Chaichian *et al* 1990, Floreanini *et al* 1991).

On the other hand, it was observed that, contrary to what happens for ordinary boson or fermion operators in the $q \to 1$ limit, the q-deformed creation operators do not transform covariantly under the dual matrix pseudogroup $SU_q(2)$. One has therefore to introduce a second type of q-boson (or q-fermion) operators $A_i^\dagger$, A_i, $i = 1, 2$, which may be obtained by using their transformation properties under $SU_q(2)$ (Pusz and Woronowicz 1989, Chaichian *et al* 1991, Nomura 1991b, c) or under the corresponding q-algebra $su_q(2)$ (Biedenharn 1990, Biedenharn and Tarlini 1990). It appears that $SU_q(2)$-covariance requires a particular coupling of both modes that is related to the non-commutativity factors in $SU_q(2)$-covariant differential calculus (Wess and Zumino 1990). Connections with the quantum group braid matrix have also been recently stressed (Rittenberg and Scheunert 1992, Hadjiivanov *et al* 1992).

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0305-4470/93/060299+08$07.50

The construction of covariant spinors $(A_1^\dagger, A_2^\dagger)$ or contravariant ones (A_1, A_2) is of course a special case of the more general problem of defining tensor operators for quantum groups (Biedenharn 1990, Biedenharn and Tarlini 1990, Nomura 1990, 1991a, b, c, Zhang *et al* 1991, Rittenberg and Scheunert 1992, Hadjiivanov *et al* 1992).

Some additional problems arise when one considers m independent sets of covariant and contravariant q-bosonic spinors $\{(A_{1s}^\dagger, A_{2s}^\dagger), (A_{1s}, A_{2s})\}$, $s = 1, 2, \ldots, m$, each acting in a different q-Fock space F_s, because, except for $s = 1$, they do not transform covariantly under $SU_q(2)$ in the tensor product space $F^{(m)} \equiv F_1 \otimes F_2 \otimes \cdots \otimes F_m$. Hence, in the latter, commutativity of the various sets should be relaxed. Such a difficulty in the theory of tensor operators for quantum groups is known to be related to the non-commutativity of the $su_q(2)$ Hopf algebra co-product (Rittenberg and Scheunert 1992).

Recently, Nomura (1991b, c) tried to solve this problem in the case where $m = 2$ by determining q-commutation relations for creation and annihilation operators $t_{is}^\dagger$, t_{is}, i, $s = 1$, 2, under the following restrictions: (i) invariance under the $SU_q(2)$ transformations, (ii) uniqueness of scalar and vector operators made from given sets of spinors (as in the $q \to 1$ limit), and (iii) consistency conditions imposed by associativity. It turns out, however, that the operators found do not satisfy property (iii) if t_{is} is defined as the Hermitian conjugate of $t_{is}^\dagger$.†

The purpose of the present letter is two-fold: firstly to show that Nomura's problem does only have a solution provided no Hermiticity relation is assumed between covariant and contravariant spinors; secondly to find by q-algebraic techniques the explicit expressions of the latter in terms of the operators $A_{is}^\dagger$, A_{is}, or $a_{is}^\dagger$, a_{is}, acting in F_s only. In such a way, we shall unify the approaches based on the transformation properties of spinors under $SU_q(2)$ and $su_q(2)$ respectively, while providing a generalization of Nomura's results to arbitrary m values and to q-fermionic spinors. Extension of the present work to $SU_q(n)$ is in progress and will be reported elsewhere.

Let us first consider the case of q-bosonic spinors. The two-dimensional matrices belonging to $SU_q(2)$ are denoted by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

where a, b, c, d are non-commuting objects satisfying the relations

$$\begin{aligned} &ba = q^{1/2}ab \qquad ca = q^{1/2}ac \qquad cb = bc \qquad db = q^{1/2}bd \\ &dc = q^{1/2}cd \qquad ad - q^{-1/2}bc = da - q^{1/2}bc = 1 \end{aligned} \tag{2}$$

and q is a positive number ($q \neq 1$). Under the *-involution, M is transformed into

$$M^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & -q^{-1/2}c \\ -q^{1/2}b & a \end{pmatrix} \tag{3}$$

† In Nomura's papers, $t_{11}^\dagger$, $t_{21}^\dagger$, $t_{12}^\dagger$ and $t_{22}^\dagger$ are denoted by $a^\dagger$, $b^\dagger$, $e^\dagger$ and $f^\dagger$, respectively. For their q-commutation relations, four different solutions are obtained. For that given in equation (2.17) of Nomura (1991c), it is easy to check that the braiding relation is not satisfied by $e^\dagger a^\dagger e$, for instance. One gets similar results for the remaining three solutions.

and q is left unchanged.

By definition, covariant and contravariant spinors, denoted by $\boldsymbol{t}_s^\dagger = (t_{1s}^\dagger, t_{2s}^\dagger)$ and $\boldsymbol{u}_s = (u_{1s}, u_{2s})$ respectively, obey the following transformation laws under $SU_q(2)$:

$$\boldsymbol{t}_s'^\dagger = \boldsymbol{t}_s^\dagger \mathsf{M} \qquad \boldsymbol{u}_s' = \boldsymbol{u}_s \mathsf{M}^* . \tag{4}$$

Here u_{is} is not assumed *a priori* to be the Hermitian conjugate of $t_{is}^\dagger$. If $\boldsymbol{u}_s = (u_{1s}, u_{2s})$ is a contravariant spinor, then $\tilde{\boldsymbol{u}}_s = (\tilde{u}_{1s}, \tilde{u}_{2s}) = (q^{1/4}u_{2s}, -q^{-1/4}u_{1s})$ is a covariant one. We shall collectively denote both covariant spinors $\boldsymbol{t}_s^\dagger$ and $\tilde{\boldsymbol{u}}_s$ by $\boldsymbol{\tau}_s$ and consider two s values (hence $m = 2$).

For a given s value, Nomura (1991b) found two sets of q-commutation relations satisfying conditions (i), (ii), and (iii), as enumerated in the introduction. One of them is given by

$$\begin{aligned}
&t_{2s}^\dagger t_{1s}^\dagger = q^{1/2} t_{1s}^\dagger t_{2s}^\dagger \qquad u_{1s}u_{2s} = q^{1/2}u_{2s}u_{1s} \qquad u_{2s}t_{1s}^\dagger = q^{1/2}t_{1s}^\dagger u_{2s} \\
&u_{1s}t_{2s}^\dagger = q^{1/2}t_{2s}^\dagger u_{1s} \qquad u_{1s}t_{1s}^\dagger = qt_{1s}^\dagger u_{1s} + 1 \\
&u_{2s}t_{2s}^\dagger = qt_{2s}^\dagger u_{2s} + (q-1)t_{1s}^\dagger u_{1s} + 1
\end{aligned} \tag{5}$$

while the other can be obtained from (5) by the substitutions $t_{1s}^\dagger \leftrightarrow t_{2s}^\dagger$, $u_{1s} \leftrightarrow u_{2s}$, and $q \to q^{-1}$. Both solutions were found previously by Pusz and Woronowicz (1989). In the following, we shall restrict ourselves to the case (5).

Considering now $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ and imposing condition (ii) of Nomura, we obtain four sets of relations of the type

$$[\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_1]_0^0 = -q^\alpha [\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2]_0^0 \qquad [\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_1]_\mu^1 = q^\beta [\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2]_\mu^1 \tag{6}$$

according to the choice made for $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$. Here $[\boldsymbol{\tau}_s \times \boldsymbol{\tau}_t]_\mu^\lambda$ denotes the coupling of the two spinors $\boldsymbol{\tau}_s$ and $\boldsymbol{\tau}_t$ to a resultant scalar or vector by means of the $SU_q(2)$ Wigner coefficients

$$\begin{aligned}
&\left\langle \tfrac{1}{2}\, m, \tfrac{1}{2} -m \mid 0\,0 \right\rangle_q = (-1)^{\frac{1}{2}-m} \frac{q^{m/2}}{\sqrt{[2]_q}} \\
&\left\langle \tfrac{1}{2} \pm \tfrac{1}{2}, \tfrac{1}{2} \pm \tfrac{1}{2} \mid 1 \pm 1 \right\rangle_q = 1 \qquad \left\langle \tfrac{1}{2}\, m, \tfrac{1}{2} -m \mid 1\,0 \right\rangle_q = \frac{q^{-m/2}}{\sqrt{[2]_q}}
\end{aligned} \tag{7}$$

where

$$[x]_q \equiv \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \tag{8}$$

is a q-number. The exponents α and β are two real numbers subject to the restriction that $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ should satisfy some consistency conditions imposed by associativity (Nomura's condition (iii)). In principle, we may have different sets $\{\alpha_a, \beta_a\}$ for the four possible choices for $\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$.

Considering first the case $\{\tau_1,\tau_2\} = \{t_1^\dagger, t_2^\dagger\}$ and working out the braiding relations for $t_{i1}^\dagger t_{j2}^\dagger t_{k1}^\dagger$ and $t_{i2}^\dagger t_{j1}^\dagger t_{k2}^\dagger$, we obtain two families of solutions for $\{\alpha_1,\beta_1\}$ corresponding to arbitrary α_1 and to $\beta_1 = \alpha_1 + \delta_1$, with $\delta_1 = 1$ or -1. Proceeding in the same way for $\{\tau_1,\tau_2\} = \{\tilde{u}_1,\tilde{u}_2\}$, $\{\tilde{u}_1, t_2^\dagger\}$, and $\{t_1^\dagger, \tilde{u}_2\}$, we get similar results, namely $\beta_a = \alpha_a + \delta_a$, $a = 2, 3, 4$ with arbitrary α_a and $\delta_a = \pm 1$. Working out finally the braiding relations for $t_{i1}^\dagger t_{j2}^\dagger \tilde{u}_{k1}$, $t_{i2}^\dagger t_{j1}^\dagger \tilde{u}_{k2}$, $t_{i1}^\dagger \tilde{u}_{j2}\tilde{u}_{k1}$, and $t_{i2}^\dagger \tilde{u}_{j1}\tilde{u}_{k2}$, we obtain the additional conditions $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta$, $\alpha_1 = \alpha_2 = \alpha$, and $\alpha_3 = \alpha_4 = -\alpha - \frac{3}{2}\delta$. So altogether we are left with two families of solutions corresponding to $\delta = 1$ or -1, and arbitrary α.

The simplest solutions correspond to the case where we have the same exponents in (6) for any $\{\tau_1,\tau_2\}$, namely $\alpha = -\frac{3}{4}\delta$ and $\beta = \frac{1}{4}\delta$. Hence the q-commutation relations for the components of two covariant q-bosonic tensors with $s = 1$ and $t = 2$, respectively, can be written as

$$\begin{aligned} &\tau_{1s}\tau_{1t} = q^{-1/4}\tau_{1t}\tau_{1s} \qquad \tau_{1s}\tau_{2t} = q^{1/4}\tau_{2t}\tau_{1s} - \left(q^{3/4} - q^{-1/4}\right)\tau_{1t}\tau_{2s} \\ &\tau_{2s}\tau_{1t} = q^{1/4}\tau_{1t}\tau_{2s} \qquad \tau_{2s}\tau_{2t} = q^{-1/4}\tau_{2t}\tau_{2s} \end{aligned} \tag{9}$$

or

$$\begin{aligned} &\tau_{1s}\tau_{1t} = q^{1/4}\tau_{1t}\tau_{1s} \qquad \tau_{1s}\tau_{2t} = q^{-1/4}\tau_{2t}\tau_{1s} \\ &\tau_{2s}\tau_{1t} = q^{-1/4}\tau_{1t}\tau_{2s} + \left(q^{1/4} - q^{-3/4}\right)\tau_{2t}\tau_{1s} \qquad \tau_{2s}\tau_{2t} = q^{1/4}\tau_{2t}\tau_{2s} \end{aligned} \tag{10}$$

according to whether $\delta = 1$ or -1. Equation (9) is reminiscent of some relations previously found by Carow-Watamura *et al* (1990). It is now clear that neither (9) nor (10) are compatible with the Hermiticity conditions $u_{is} = (t_{is}^\dagger)^\dagger$ imposed by Nomura (1991b, c). From (9), for instance, we indeed obtain $t_{11}^\dagger t_{22}^\dagger = q^{1/4} t_{22}^\dagger t_{11}^\dagger - (q^{3/4} - q^{-1/4}) t_{12}^\dagger t_{21}^\dagger$ and $u_{11}u_{22} = q^{1/4}u_{22}u_{11}$. Similar conclusions would have been drawn had we left α arbitrary. As a matter of fact, the existence of a pair of solutions, as given in (9) and (10), is intimately connected with the lack of Hermiticity relation between $t_{is}^\dagger$ and u_{is}. It can indeed easily be checked that if $\{t_1^\dagger, t_2^\dagger\}$, $\{t_1^\dagger, \tilde{u}_2\}$, $\{\tilde{u}_1, t_2^\dagger\}$, $\{\tilde{u}_1, \tilde{u}_2\}$ satisfy (9), then the Hermitian conjugates $\{u_1^\dagger, u_2^\dagger\}$, $\{u_1^\dagger, \tilde{t}_2\}$, $\{\tilde{t}_1, u_2^\dagger\}$, $\{\tilde{t}_1, \tilde{t}_2\}$ fulfil (10). We shall now contrast the approach based upon $SU_q(2)$ considered above with that relying on $su_q(2)$. With the help of the latter, which is much simpler, we shall derive the q-commutation relations of m sets of q-bosonic covariant spinors τ_s, $s = 1, \ldots, m$, where τ_s stands for $t_s^\dagger = (t_{1s}^\dagger, t_{2s}^\dagger)$ or $\tilde{u}_s = (\tilde{u}_{1s}, \tilde{u}_{2s})$.

The $su_q(2)$ q-algebra is the associative algebra with generators J_0, J_+, J_-, and relations

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0]_q \tag{11}$$

where $[2J_0]_q$ is defined as in (8). It is a quasitriangular Hopf algebra with a coproduct Δ, a co-unit ϵ, an antipode S, and a universal $\mathcal{R}$ matrix, given by

$$\Delta J_0 = J_0 \otimes I + I \otimes J_0 \qquad \Delta J_\pm = J_\pm \otimes q^{J_0/2} + q^{-J_0/2} \otimes J_\pm \tag{12a}$$

$$\epsilon J_0 = \epsilon J_\pm = 0 \tag{12b}$$

$$SJ_0 = -J_0 \qquad SJ_\pm = -q^{\pm 1/2} J_\pm \tag{12c}$$

$$\mathcal{R} = q^{J_0 \otimes J_0} \sum_{n=0}^{\infty} \frac{(1-q^{-1})^n}{[n]_q!} q^{n(n-1)/4} (q^{J_0/2} J_+)^n \otimes (q^{-J_0/2} J_-)^n \tag{12d}$$

respectively (Majid 1990).

The q-algebra $su_q(2)$ admits a q-analogue of $su(2)$ Schwinger realization (Biedenharn 1989, Macfarlane 1989)

$$J_0 = \tfrac{1}{2}(N_1 - N_2) \qquad J_+ = a_1^\dagger a_2 \qquad J_- = a_2^\dagger a_1 \tag{13}$$

in terms of the q-boson operators $N_i = (N_i)^\dagger$, $a_i^\dagger$, $a_i = (a_i^\dagger)^\dagger$, $i = 1, 2$, acting in a q-Fock space F and satisfying the relations

$$\begin{aligned} &[N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger \qquad [N_i, a_j] = -\delta_{ij} a_j \\ &[a_1^\dagger, a_2^\dagger] = [a_1, a_2] = [a_1, a_2^\dagger] = [a_2, a_1^\dagger] = 0 \\ &a_i a_i^\dagger - q^{\pm 1/2} a_i^\dagger a_i = q^{\mp N_i/2} . \end{aligned} \tag{14}$$

An irreducible tensor of rank λ with respect to $su_q(2)$ is defined as a set of $2\lambda + 1$ operators T_μ^λ, $\mu = \lambda, \lambda - 1, \ldots, -\lambda$, satisfying the relations (Rittenberg and Scheunert 1992)

$$J_0(T_\mu^\lambda) = \mu\, T_\mu^\lambda \qquad J_\pm(T_\mu^\lambda) = \left([\lambda \mp \mu]_q [\lambda \pm \mu + 1]_q\right)^{1/2} T_{\mu\pm1}^\lambda \tag{15}$$

where

$$J_\sigma(T_\mu^\lambda) \equiv \sum_r (J_\sigma)_r^1\, T_\mu^\lambda S(J_\sigma)_r^2 \qquad (\sigma = +, 0, -) \tag{16}$$

and $\Delta J_\sigma = \sum_r (J_\sigma)_r^1 \otimes (J_\sigma)_r^2$. Hence, by virtue of (12),

$$J_0(T_\mu^\lambda) \equiv [J_0, T_\mu^\lambda] \qquad J_\pm(T_\mu^\lambda) \equiv J_\pm T_\mu^\lambda q^{-J_0/2} - q^{\pm 1/2} q^{-J_0/2} T_\mu^\lambda J_\pm . \tag{17}$$

From T_μ^λ, one can construct by Hermitian conjugation another irreducible tensor of rank λ

$$\tilde{T}_\mu^\lambda = (-1)^{\lambda-\mu} q^{\mu/2} (T_{-\mu}^\lambda)^\dagger . \tag{18}$$

Covariant q-boson creation (resp. annihilation) operators $A_i^\dagger$ (resp. $\tilde{A}_i$), $i = 1, 2$, acting in F, are defined by (15) where $\lambda = \pm\mu = \frac{1}{2}$, $T_{1/2}^{1/2} = A_1^\dagger$ (resp. $\tilde{A}_1$), and $T_{-1/2}^{1/2} = A_2^\dagger$ (resp. $\tilde{A}_2$). Up to some arbitrary function of $N_1 + N_2$, they can be expressed in terms of N_i, $a_i^\dagger$, a_i as†

$$\begin{aligned} &A_1^\dagger = a_1^\dagger q^{N_1/4} \qquad A_2^\dagger = a_2^\dagger q^{(2N_1+N_2)/4} \\ &\tilde{A}_1 = q^{1/4} A_2 = q^{(2N_1+N_2+1)/4} a_2 \qquad \tilde{A}_2 = -q^{-1/4} A_1 = -q^{(N_1-1)/4} a_1 . \end{aligned} \tag{19}$$

† The choice made in (19) corresponds to that of Pusz and Woronowicz (1989), but differs from that of Biedenharn (1989) and of Rittenberg and Scheunert (1992). Note that only the former leads to simple q-commutation relations.

With the contravariant annihilation operators A_i, the covariant creation operators $A_i^\dagger$ satisfy q-commutation relations analogous to those given in (5) with $A_i^\dagger$ and A_i substituted for $t_{is}^\dagger$ and u_{is} respectively.

Consider now m independent q-Fock spaces F_s, $s = 1, 2, \ldots, m$, and corresponding q-boson operators N_{is}, $a_{is}^\dagger$, a_{is}, $A_{is}^\dagger$, A_{is}, $i = 1, 2$, $s = 1, 2, \ldots, m$. For different s values, these operators are assumed to commute with one another, while for a given s value they satisfy equations similar to (5) and (14).

In the tensor product space $F^{(m)}$, the action of $su_q(2)$ is given by the iterated co-product

$$\Delta^{(m-1)} J_\sigma \equiv \begin{cases} \Delta J_\sigma & \text{if } m = 2 \\ (\Delta \otimes I^{(m-2)}) \Delta^{(m-2)} J_\sigma & \text{if } m > 2 \end{cases} \tag{20}$$

or

$$\begin{aligned} \Delta^{(m-1)} J_0 &= \Delta^{(m-2)} J_0 \otimes I + I^{(m-1)} \otimes J_0 \\ \Delta^{(m-1)} J_\pm &= \Delta^{(m-2)} J_\pm \otimes q^{J_0/2} + q^{-\Delta^{(m-2)} J_0/2} \otimes J_\pm \end{aligned} \tag{21}$$

where $I^{(m)} \equiv I \otimes I \otimes \cdots \otimes I$ (m times). Hence, irreducible tensors of rank λ in $F^{(m)}$ are defined by (15) and (17), where J_σ is replaced by $\Delta^{(m-1)} J_\sigma$.

As emphasized by Rittenberg and Scheunert (1992), if T_μ^λ is an irreducible tensor of rank λ in F_1 or in F_2, then $T_\mu^\lambda \otimes I$ is also an irreducible tensor of rank λ in $F^{(2)} = F_1 \otimes F_2$ in the former case, whereas $I \otimes T_\mu^\lambda$ is not in the latter. By generalizing their procedure for constructing irreducible tensors in the tensor product space $F^{(2)}$, the following statements can be easily proved:

(i) If T_μ^λ is an irreducible tensor of rank λ in $F^{(m-1)}$, then $T_\mu^\lambda \otimes I$ has the same property in $F^{(m)}$.

(ii) If T_μ^λ is an irreducible tensor of rank λ in F_m, then

$$\mathcal{R}_{m,m-1} \mathcal{R}_{m,m-2} \ldots \mathcal{R}_{m1} (I^{(m-1)} \otimes T_\mu^\lambda) \mathcal{R}_{m1}^{-1} \ldots \mathcal{R}_{m,m-2}^{-1} \mathcal{R}_{m,m-1}^{-1} \tag{22}$$

has the same property in $F^{(m)}$. In (22), $\mathcal{R}_{ms}$, $s = 1, \ldots, m-1$, is defined by

$$\mathcal{R}_{ms} = \sum_r I^{(s-1)} \otimes \mathcal{R}_r^2 \otimes I^{(m-s-1)} \otimes \mathcal{R}_r^1 \tag{23}$$

where $\mathcal{R}_r^1$ and $\mathcal{R}_r^2$ are given by the decomposition $\mathcal{R} = \sum_r \mathcal{R}_r^1 \otimes \mathcal{R}_r^2$ of the $\mathcal{R}$ matrix (12d).

By using (12d), (22), and (23), one obtains the result that the operators

$$t_{11}^\dagger = A_{11}^\dagger \otimes I^{(m-1)} \qquad t_{21}^\dagger = A_{21}^\dagger \otimes I^{(m-1)} \tag{24}$$

and

$$\begin{aligned} t_{1s}^\dagger &= q^{\Delta^{(s-2)} J_0/2} \otimes A_{1s}^\dagger \otimes I^{(m-s)} \\ t_{2s}^\dagger &= q^{-\Delta^{(s-2)} J_0/2} \otimes A_{2s}^\dagger \otimes I^{(m-s)} + (q^{1/4} - q^{-3/4}) \Delta^{(s-2)} J_- \otimes A_{1s}^\dagger \otimes I^{(m-s)} \end{aligned} \tag{25}$$

where $s = 2, \ldots, m$, are the components of m covariant spinors in $F^{(m)}$. The same is true for the operators $\tilde{u}_{is}$, $i = 1, 2$, $s = 1, 2, \ldots, m$, obtained by replacing $A^\dagger_{is}$ by $\tilde{A}_{is}$ in (24) and (25). It is actually straightforward to check that all these operators satisfy (15) and (17) where J_σ is replaced by $\Delta^{(m-1)}J_\sigma$.

It is now an easy task to prove that the operators defined in (24), (25) and those obtained by substituting $\tilde{A}_{is}$ for $A^\dagger_{is}$ have the same q-commutation relations as those obtained by using transformations under $SU_q(2)$, namely that they satisfy (5) and (9), where in the former $1 \leqslant s \leqslant m$ and in the latter $1 \leqslant s < t \leqslant m$. With the help of (13), (19) and (21), we have thus realized the operators that are solutions of Nomura's problem in terms of N_{is}, $A^\dagger_{is}$, A_{is} or equivalently in terms of N_{is}, $a^\dagger_{is}$, a_{is}. From (24), (25), and their counterparts for $\tilde{u}_{is}$, it is clear that $u_{i1} = (t^\dagger_{i1})^\dagger$, but $u_{is} \neq (t^\dagger_{is})^\dagger$ for $s = 2, \ldots, m$. In spite of this lack of Hermiticity, the $SU_q(2)$-invariant operators $\sum_i t^\dagger_{is}u_{is}$, $s = 1, \ldots, m$, are Hermitian as they can be expressed as

$$\sum_i t^\dagger_{is}u_{is} = I^{(s-1)} \otimes \sum_i A^\dagger_{is}A_{is} \otimes I^{(m-s)}. \tag{26}$$

The case of q-fermionic spinors can be dealt with in a similar way. The q-algebra $su_q(2)$ admits the realization (13) in terms of q-fermion operators N_i, $a^\dagger_i$, a_i, $i = 1, 2$, satisfying relations of the type

$$\begin{aligned} &[N_i, a^\dagger_j] = \delta_{ij}a^\dagger_j \qquad [N_i, a_j] = -\delta_{ij}a_j \\ &\{a^\dagger_1, a^\dagger_2\} = \{a_1, a_2\} = \{a_1, a^\dagger_2\} = \{a_2, a^\dagger_1\} = 0 \\ &a_i a^\dagger_i + q^{\pm 1/2}a^\dagger_i a_i = q^{\pm N_i/2}. \end{aligned} \tag{27}$$

Such operators may, however, be considered as ordinary fermion operators (Floreanini *et al* 1991).

Covariant q-fermionic spinors are given by equations similar to (24), (25), and their counterparts for $\tilde{u}_{is}$, where $A^\dagger_{is}$ now assumes the following form:

$$A^\dagger_{1s} = a^\dagger_{1s}q^{-N_{1s}/4} = a^\dagger_{1s} \qquad A^\dagger_{2s} = a^\dagger_{2s}q^{-(2N_{1s}+N_{2s})/4} = a^\dagger_{2s}(1 + (q^{-1/2} - 1)N_{1s}) \tag{28}$$

where use is made of the property $N^2_{is} = N_{is}$. Instead of (5) and (9), they satisfy q-anticommutation relations of the type

$$\begin{aligned} &(t^\dagger_{1s})^2 = (t^\dagger_{2s})^2 = (u_{1s})^2 = (u_{2s})^2 \qquad t^\dagger_{2s}t^\dagger_{1s} = -q^{-1/2}t^\dagger_{1s}t^\dagger_{2s} \\ &u_{1s}u_{2s} = -q^{-1/2}u_{2s}u_{1s} \qquad u_{2s}t^\dagger_{1s} = -q^{-1/2}t^\dagger_{1s}u_{2s} \qquad u_{1s}t^\dagger_{2s} = -q^{-1/2}t^\dagger_{2s}u_{1s} \\ &u_{1s}t^\dagger_{1s} = -t^\dagger_{1s}u_{1s} + 1 \qquad u_{2s}t^\dagger_{2s} = -t^\dagger_{2s}u_{2s} + (q^{-1} - 1)t^\dagger_{1s}u_{1s} + 1 \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\tau_{1s}\tau_{1t} = -q^{-1/4}\tau_{1t}\tau_{1s} \qquad \tau_{1s}\tau_{2t} = -q^{1/4}\tau_{2t}\tau_{1s} + (q^{3/4} - q^{-1/4})\tau_{1t}\tau_{2s} \\ &\tau_{2s}\tau_{1t} = -q^{1/4}\tau_{1t}\tau_{2s} \qquad \tau_{2s}\tau_{2t} = -q^{-1/4}\tau_{2t}\tau_{2s} \end{aligned} \tag{30}$$

where $s < t$.

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